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Problem Set #2 Solutions

Answer 1:
The stick contracts according to the Lorentz contraction:

$$\begin{aligned} L' &= L/\gamma \\ 0.1 &= 1/\gamma \\ \gamma &= 10 \end{aligned}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \quad \rightarrow \quad v^2/c^2 = 1 - 1/\gamma^2$$

$$\begin{aligned} v/c &= \sqrt{1 - 1/\gamma^2} \\ v &= 0.995c \end{aligned}$$

Answer 2:
In order to make an average revolutions of $N = 10^6$, they should live for:

$$T = \frac{N 2\pi R}{v}$$

where $R = 100$ m is the radius, and v is the speed of the μ particles. Then, its life in the lab-frame must be:

$$T = \gamma\tau$$

where $\tau = 2.2 \times 10^{-6}$ sec is the proper lifetime of the muons. Using $\beta = v/c$, $v = \beta c$, we write:

$$T = \frac{N 2\pi R}{\beta c} = \gamma\tau$$

Let us define the dimensionless number a:

$$a \equiv \frac{N 2\pi R}{c\tau} = 0.95 \times 10^6$$

Then, we have:

$$\begin{aligned} \beta\gamma &= a \\ \beta^2\gamma^2 &= a^2 \\ \beta^2 \frac{1}{1-\beta^2} &= a^2 \\ \beta^2 &= a^2 - a^2\beta^2 \\ \beta &= \frac{1}{\sqrt{1 + 1/a^2}} \\ \beta &\approx 1 - \frac{1}{2a^2} \\ \beta &\approx 1 \\ v &\approx c \end{aligned}$$

Note that $\beta\gamma = a \;\Rightarrow\; \gamma = a/\beta \;\Rightarrow\; \gamma \approx 10^6$ which indicates a total energy of $E = \gamma m_\mu c^2 \approx 100$ TeV, where $m_\mu \approx 100$ MeV$/c^2$. 100 TeV is a huge energy for a particle accelerator. (Tevatron of Fermilab/USA can accelerate protons upto a TeV.)

Answer 3:

The driver approaches to the source, so our relativistic Doppler formula reads:

$$f' = \gamma f(1 + v/c)$$

we use $\lambda = c/f = 650$ nm, and $\lambda' = c/f' = 530$ nm, with $\lambda/\lambda' = 1.226$:

$$\begin{aligned}
\frac{c}{\lambda'} &= \frac{c}{\lambda}\gamma(1+\beta) \\
\frac{\lambda}{\lambda'} &= \gamma(1+\beta) \\
&= \frac{1+\beta}{\sqrt{1-\beta^2}} \\
&= \frac{\sqrt{1+\beta}\sqrt{1+\beta}}{\sqrt{1-\beta}\sqrt{1+\beta}} \\
&= \frac{\sqrt{1+\beta}}{\sqrt{1-\beta}} \\
\frac{\lambda^2}{\lambda'^2} &= \frac{1+\beta}{1-\beta}
\end{aligned}$$

solve it for beta:

$$\beta = \frac{1-(\lambda/\lambda')^2}{1+(\lambda/\lambda')^2} = \frac{1-1.226^2}{1+1.226^2} = 0.2$$
$$v = 0.2c$$

Answer 4:

Say, the speed of the particle in the S–frame is u, then we write the speed in S'–frame, u', where the S'–frame moves with speed v with respect to the S–frame[1]:

$$u' = \frac{u-v}{1-uv}$$

Then, we drive γ in S'–frame using the above u': $\gamma(u') = 1/\sqrt{1-u'^2}$:

$$\begin{aligned}
1-u'^2 &= 1-\frac{(u-v)^2}{(1-uv)^2} \\
&= \frac{(1-uv)^2-(u-v)^2}{(1-uv)^2} \\
&= \frac{1-2uv+u^2v^2-u^2+2uv-v^2}{(1-uv)^2} \\
&= \frac{1-u^2-v^2+v^2u^2}{(1-uv)^2} \\
&= \frac{(1-v^2)(1-u^2)}{(1-uv)^2} \\
\gamma(u') = 1/\sqrt{1-u'^2} &= \frac{1-uv}{\sqrt{1-v^2}\sqrt{1-u^2}} \\
\gamma(u') &= \gamma(v)\cdot\gamma(u)\cdot(1-uv)
\end{aligned}$$

Now, let's write energy in S'–frame:

$$\begin{aligned}
E' &= \gamma(u')\cdot m \\
&= \gamma(v)\cdot\gamma(u)\cdot(1-uv)\cdot m \\
&= \gamma(v)\left[(m\cdot\gamma(u))-(m\cdot\gamma(u)\cdot u)\,v\right]
\end{aligned}$$

where $m\cdot\gamma(u) = E$, and $m\cdot\gamma(u)\cdot u = p$. Thus, we get:

$$E' = \gamma(v)(E-pv) \tag{1}$$

[1] You can assume $c = 1$ for this question to simplify the equations.

Similarly, the momentum in S'–frame can be derived as:

$$\begin{aligned} p' &= \gamma(u') \cdot m \cdot u' \\ &= \gamma(v) \cdot \gamma(u) \cdot (1 - uv) \cdot m \cdot \frac{u - v}{1 - uv} \\ &= \gamma(v) \cdot \gamma(u) \cdot (u - v) \cdot m \\ &= \gamma(v) \left[(m \cdot \gamma(u) \cdot u) - (m \cdot \gamma(u)) \cdot v \right] \\ p' &= \gamma(v)(p - Ev) \end{aligned} \tag{2}$$

Let's write the Lorentz transformations for completeness:

$$x' = \gamma(v) \cdot (x - vt) \tag{3}$$

$$t' = \gamma(v) \cdot (t - xv) \tag{4}$$

Finally, we write the quantity $px - Et$ in S'–frame using the equations (1), (2), (3), and (4):

$$\begin{aligned} p'x' - E't' &= \{\gamma(v)(p - Ev)\}\{\gamma(v) \cdot (x - vt)\} - \{\gamma(v)(E - pv)\}\{\gamma(v) \cdot (t - xv)\} \\ &= \gamma(v)^2 \cdot \left[px - pvt - Evx + Ev^2t - Et + Evx + pvt - pxv^2 \right] \\ &= \frac{1}{1 - v^2} \cdot \left[px(1 - v^2) - Et(1 - v^2) \right] \\ &= \frac{1}{1 - v^2} \cdot (1 - v^2) \left[px - Et \right] \\ p'x' - E't' &= px - Et \end{aligned}$$

Thus, this quantity is invariant.

Answer 5:
The total energy of the electron is ($m_e \approx 0.5$ MeV$/c^2$):

$$\begin{aligned} E_{e^-} &= E_K + m_e c^2 \\ &= 1 + 0.5 \text{ MeV} \\ &= 1.5 \text{ MeV} \end{aligned}$$

The energy of the positron is equal to only its mass energy, which is $E_{e^+} = m_e c^2 = 0.5$ MeV:

$$E_{e^-} + E_{e^+} = E_1 + E_2 = 2 \text{ MeV} \tag{5}$$

Where $E_{1,2}$ are the energies of the outgoing two photons.

Assume that the direction of the electron momentum is $+\hat{x}$. Then, since its known that one of the photon goes to the $+\hat{x}$ direction, the other photon should go to either $+\hat{x}$ or $-\hat{x}$ direction; there is no y component of the momentum.

The momentum of the electron is (we use $E^2 = p^2c^2 + m^2c^4$)

$$p_{e^-} = \sqrt{E_{e^-}^2 - m_e^2 c^4}/c = \sqrt{1.5^2 - 0.5^2} = \sqrt{2} \text{ MeV/c}$$

The momentum of the positron is $p_{e^+} = 0$; it is at rest. So, the total momentum before the collision is $p = p_{e^-} + p_{e^+} = \sqrt{2}$ MeV/c.

For a photon, we have $E^2 = p^2c^2 + m_\gamma^2 c^4$. But, $m_\gamma = 0$, which gives us the simpler relation $p = E/c$. Then we use the momentum conservation to get our second equation. For this we have two options depending on the direction of the second photon. Let us assume it goes to $+\hat{x}$ direction. Then:

$$\begin{aligned} p &= p_1 + p_2 \\ \sqrt{2} \text{ MeV/c} &= (E_1 + E_2)/c \\ E_1 + E_2 &= \sqrt{2} \text{ MeV} \end{aligned}$$

If we check our energy conversation in Eqn. (5), we conclude that $+\hat{x}$ cannot be correct: Thus, the second photon must go to the opposite direction, $-\hat{x}$. Then, we rewrite the momentum conservation formula as:

$$\begin{aligned} p &= p_1 - p_2 \\ \sqrt{2}\,\mathrm{MeV/c} &= (E_1 - E_2)/c \\ E_1 - E_2 &= \sqrt{2}\,\mathrm{MeV} \end{aligned} \tag{6}$$

We can solve E_1 and E_2 using Eqn. (5) and (6):

$$\begin{aligned} E_1 &= 1.707\ \mathrm{MeV} \\ E_2 &= 0.293\ \mathrm{MeV} \end{aligned}$$