Problem Set #3 Solutions

Answer 1:
The total energy of the particle is $E = \gamma mc^2$ where $\gamma = (1-\beta^2)^{-1/2}$, and $\beta = 0.5$. Then:

$$E = \gamma mc^2 = \frac{(0.001\ \text{kg})\ \times\ (3\times 10^8\ \text{m/s})^2}{\sqrt{1-0.5^2}} \approx 10^{14}\ \text{J}$$

The total energy put by the dam is:

$$\begin{aligned} E &= \text{Power} \times t \\ 10^{14}\ \text{J} &= (2400\times 10^6\ \text{W}) \times t \\ t &= 41667\ \text{sec} \end{aligned}$$

Thus, the Atatürk Dam produces 10^{14} joules of energy in about 12 hours.

Answer 2:
The total mass of 1 gr of matter and 1 gr of antimatter is 2 grams. Then, the total energy released by the annihilation of these masses is simply $E_{tot} = m_{tot}\, c^2$. On the other hand, lifting mass M to a height of $h = 1$ km requires a potential energy of $E = Mgh$. Thus, we get:

$$\begin{aligned} m_{\text{tot}}\, c^2 &= Mgh \\ M &= \frac{m_{\text{tot}}\, c^2}{gh} \\ M &= \frac{(0.002\ \text{kg})\times(3\times 10^8\ \text{m/s})^2}{(9.8\ \text{m/s}^2)\times(1000\ \text{m})} \\ M &\approx 2\times 10^{10}\ \text{kg} \end{aligned}$$

Note that this is roughly two orders of magnitude greater than the weight of an aircraft carrier.

Answer 3:
The total energy and momentum of the system before the decay is:

$$E = 625\ \text{MeV} \qquad \vec{p} = \vec{0}$$

The momentum cooonservation yields:

$$\begin{aligned} \vec{p}_{tot} &= \vec{p}_{\pi^+} + \vec{p}_{\pi^-} = \vec{0} \\ \vec{p}_{\pi^+} &= -\vec{p}_{\pi^-} \end{aligned}$$

Which means that the magnitudes of the momentums are the same, but the directions are opposite to each other. The energy conservation gives (taking $c = 1$):

$$\begin{aligned} E_{\text{tot}} &= E_{\pi^+} + E_{\pi^-} \\ E_{\text{tot}} &= \sqrt{p_\pi^2 + m_\pi^2} + \sqrt{p_\pi^2 + m_\pi^2} = 2\sqrt{p_\pi^2 + m_\pi^2} \\ (E_{\text{tot}}/2)^2 &= p_\pi^2 + m_\pi^2 \\ p_\pi &= \sqrt{E_{\text{tot}}^2/4 - m_\pi^2} = \sqrt{625^2/4 - 140^2}\ \text{MeV/c} \\ p_\pi &= 279\ \text{MeV/c} \end{aligned}$$

Thus, each pion will have a momentum of 279 MeV/c, moving to the **opposite** directions.

Answer 4:
We solve this using 4-momentums (take $c = 1$):

$$P_p^\nu + P_d^\nu = P_{^3\text{He}}^\nu + P_\gamma^\nu$$

Since, $P^\nu_{^3\text{He}}$ is the only unknown and its square is the invariant quantity of $m^2 c^4$, where m is the mass of ^{3}He, we simply leave this term alone at one side of the equation and take the square of both sides. First, let us write the remaining terms:

$$\begin{aligned} P^\nu_p &= (E_p, \vec{p}_p) = (m_p, \vec{0}) \\ P^\nu_d &= (E_d, \vec{p}_d) = (m_d, \vec{0}) \\ P^\nu_\gamma &= (E_\gamma, \vec{p}_\gamma) = (p_\gamma, \vec{p}_\gamma) \qquad \text{where } p_\gamma = E_\gamma/c = 5.5 \text{ MeV/c} \end{aligned}$$

Then, we write:

$$\begin{aligned} P^\nu_{^3\text{He}} &= P^\nu_p + P^\nu_d - P^\nu_\gamma \\ P^\nu_{^3\text{He}} &= (m_p, \vec{0}) + (m_d, \vec{0}) - (p_\gamma, \vec{p}_\gamma) \\ P^\nu_{^3\text{He}} &= (m_p + m_d - p_\gamma, \vec{p}_\gamma) \\ (P^\nu_{^3\text{He}})^2 &= (m_p + m_d - p_\gamma, \vec{p}_\gamma) \cdot (m_p + m_d - p_\gamma, \vec{p}_\gamma) \\ m^2 &= (m_p + m_d - p_\gamma)^2 - p_\gamma^2 \\ m^2 &= (938.28 + 1875.6 - 5.5)^2 - 5.5^2 \\ m &= 2718 \text{ MeV/c}^2 \end{aligned}$$

which is close to the observed mass of ^{3}He.

Answer 5:

We immediately write the 4-momentum equation for this reaction (take $c = 1$):

$$P^\nu_{n_0} + P^\nu_{p_0} = P^\nu_n + P^\nu_p + P^\nu_\gamma$$

Where the following quantities are numerically known (take the direction of $\vec{p}_{n_0}$ to be $\hat{x}$):

$$\begin{aligned} P^\nu_{n_0} &= (E_{n_0}, \vec{p}_{n_0}) = (m + T_0, p_0 \hat{x}) \\ P^\nu_{p_0} &= (E_{p_0}, \vec{p}_{p_0}) = (m, 0) \\ P^\nu_p &= (E_p, \vec{p}_p) = (m + T_p, p_{p,x}\,\hat{x} + p_{p,y}\,\hat{y}) \end{aligned}$$

where

$$\begin{aligned} p_0 &= \sqrt{(m + T_0)^2 - m^2} \\ p_{p,x} &= \cos\theta_p \sqrt{(m + T_p)^2 - m^2} \\ p_{p,y} &= \sin\theta_p \sqrt{(m + T_p)^2 - m^2} \end{aligned} \qquad (1)$$

There is no information about the outgoing γ particle, and we do not need to know about it. Thus, let us leave that alone at one side of the equation as our regular trick:

$$P^\nu_\gamma = P^\nu_{n_0} + P^\nu_{p_0} - P^\nu_n - P^\nu_p \qquad (2)$$

Since we know three of the terms appearing at the right side of the equation, let us define a single 4-momentum quantity that includes all these three terms as a single quantity:

$$P^\nu \equiv P^\nu_{n_0} + P^\nu_{p_0} - P^\nu_p = (E, \vec{p})$$

where

$$E = (m + T_0) + m - (m + Tp) = m + T_0 - T_p \qquad (3)$$

$$\vec{p} = [p_0 - p_{p,x}\,,\, -p_{p,y}] \qquad (4)$$

Then, our Equation 2 can simply be written as:

$$\begin{aligned} P^\nu_\gamma &= P^\nu - P^\nu_n \\ (P^\nu_\gamma)^2 &= (P^\nu - P^\nu_n)^2 \\ m_\gamma^2 &= (P^\nu)^2 - 2P^\nu \cdot P^\nu_n + (P^\nu_n)^2 \\ 0 &= (E^2 - p^2) - 2(EE_n - \vec{p} \cdot \vec{p}_n) + m^2 \\ 0 &= (E^2 - p^2 + m^2) - 2EE_n + 2\vec{p} \cdot \vec{p}_n \\ 2\vec{p} \cdot \vec{p}_n &= 2EE_n - (E^2 - p^2 + m^2) \end{aligned}$$

Then, we can write $\vec{p}_n$ as $\hat{n}\, p_n$, where $\hat{n}$ is the unit direction vector which is given by the question:

$$\hat{n} \equiv [\cos\theta_n, \sin\theta_n] \tag{5}$$

Thus, we get an equation where p_n and E_n are the only unknowns.

$$\begin{aligned} (2\vec{p}\cdot\hat{n}_n)\, p_n &= 2EE_n - (E^2 - p^2 + m^2) \\ \frac{\vec{p}\cdot\hat{n}_n}{E}\, p_n &= E_n - \frac{E^2 - p^2 + m^2}{2E} \end{aligned}$$

Let us define the following quantities α and β (numerically known) as:

$$\alpha \equiv \frac{\vec{p}\cdot\hat{n}_n}{E} \qquad\qquad \beta \equiv \frac{E^2 - p^2 + m^2}{2E} \tag{6}$$

Then, we get:

$$\begin{aligned} \alpha\, p_n &= E_n - \beta \\ \alpha^2\, p_n^2 &= (E_n - \beta)^2 \\ \alpha^2\,(E_n^2 - m^2) &= E_n^2 - 2E_n\beta + \beta^2 \\ 0 &= (1-\alpha^2)\, E_n^2 - 2(\beta)\, E_n + (\beta^2 + \alpha^2 m^2) \end{aligned}$$

Now, let us define:

$$A \equiv 1 - \alpha^2 \qquad\qquad B \equiv \beta \qquad\qquad C \equiv \beta^2 + \alpha^2 m^2 \tag{7}$$

Then, our equation reads:

$$0 = A\, E_n^2 - 2B\, E_n + C$$

We solve for E_n, thus for T_n:

$$E_n = \frac{B \pm \sqrt{B^2 - AC}}{A} \qquad\qquad T_n = E_n - m$$

That is the solution for part a. All we need to do is putting the variables we defined in Equations 1, 3, 4, 5, 6, and 7 into this solution. Since all these variables are known, finding the solution is a pretty easy numerical/computational calculation. Also, note that when $\Delta = B^2 - AC > 0$, there are two valid solutions, and when that is less than zero there is no solution. Two-solution case becomes possible due the extra degree of freedom brought by the third product particle, the γ. That means that the γ particle will go to different directions for these two solutions. Once we know E_n we can easily extract the momentum of the γ, if we were asked.

Let us, now, draw the solution, which is known as the *kinematic locus* of this reaction. Following is a Matlab source code that performs the above operations for an array of T_p from 0 to 225 MeV, and stores the valid solutions into an array to be plotted at the end. This is a template, and can be used for any programming language, or it can be used as a guide to use a calculator-based manual drawing for a series of solutions.

```
m = 938.28;             % mass of nucleons (both for n and p)
T0 = 225;               % Kinetic energy of incoming particle
theta_p =  20/180*pi;   % angle in radian
theta_n = -20/180*pi;   % angle in radian
p0 = sqrt((m+T0)^2-m^2);
nx = cos(theta_n);           % Eqn 5a
ny = sin(theta_n);           % Eqn 5b

N = 0;    % Number of solutions

for Tp=0:225
  p_px = cos(theta_p)*sqrt((m+Tp)^2-m^2);
  p_py = sin(theta_p)*sqrt((m+Tp)^2-m^2);
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```
  E = m + T0 - Tp;                        % Eqn 3
  px = p0 - p_px;                         % Eqn 4a
  py = -p_py;                             %      b
  alpha = (px*nx+py*ny)/E;                % Eqn 6a
  beta = (E^2-(px*px+py*py)+m^2)/2/E;     %      b
  A = 1 - alpha^2;                        % Eqn 7a
  B = beta;                               %      b
  C = beta^2 + alpha^2 * m^2;             %      c
  delta = B^2 - A*C;

  % we get two solutions
  if delta>0
    % plus-solution
    N = N+1;
    tp(N) = Tp;
    tn(N) = (B + sqrt(delta))/A - m;

    % minus-solution
    N = N+1;
    tp(N) = Tp;
    tn(N) = (B - sqrt(delta))/A - m;
  end

end

plot(tp, tn, '.')
```