

Problem Set #8 Solutions

Answer 1:

We write the normalization equation:

$$1 = \int_{-\infty}^{\infty} |\psi(p)|^2 dp$$

Since, the wave function includes an absolute value function, we have to divide the integrant into two parts:

$$1 = \int_{-\infty}^0 [C e^{ap/\hbar}]^2 dp + \int_0^{\infty} [C e^{-ap/\hbar}]^2 dp$$

By symmetry, both integrals are equal to each other, thus we evaluate one of them and multiply it by two:

$$\begin{aligned} 1 &= 2 \cdot C^2 \int_0^{\infty} e^{-2ap/\hbar} dp \\ &= 2C^2 \left(-\frac{\hbar}{2a} \right) \left(e^{-2ap/\hbar} \right) \Big|_0^{\infty} \\ &= -\frac{C^2 \hbar}{a} (0 - 1) \\ C &= \sqrt{\frac{a}{\hbar}} \end{aligned}$$

Answer 2:

Note that, we derived the fourier transformation equations in class between $\psi(x) \leftrightarrow g(k)$. Using $p = \hbar k$, $dp = \hbar dk$, we can:

$$\begin{aligned} dP &= |g(k)|^2 dk \\ &= |g(p/\hbar)|^2 \frac{dp}{\hbar} \\ |\psi_p(p)|^2 dp &= |g(p/\hbar)|^2 \frac{dp}{\hbar} \end{aligned}$$

which gives us:

$$\psi_p(p) = \frac{1}{\sqrt{\hbar}} g(p/\hbar)$$

or, equivalently:

$$g(k) = \sqrt{\hbar} \psi_p(k\hbar)$$

[Note that, we ignored the phase, $e^{i\phi}$, while dealing with the magnitude function above. This is no concern for us, as it will satisfy the Schrödinger equation with or without a phase, which is a constant.]

Now, we can rewrite our Fourier transformation using momentum wave function:

$$\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2\pi}} \int g(k) e^{ikx} dk \\ \psi(x) &= \frac{1}{\sqrt{2\pi}} \int \sqrt{\hbar} \psi_p(k\hbar) e^{ipx/\hbar} \frac{dp}{\hbar} \end{aligned}$$

Thus, we get:

$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi_p(p) e^{ipx/\hbar} dp$	similarly $\rightarrow$	$\psi_p(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x) e^{-ipx/\hbar} dx$
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Now, we can perform the transformation:

$$\begin{aligned}
 \psi(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi_p(p) e^{ipx/\hbar} dp \\
 &= \frac{1}{\sqrt{2\pi\hbar}} \sqrt{\frac{a}{\hbar}} \int_{-\infty}^{\infty} e^{-a|p|/\hbar} e^{ipx/\hbar} dp \\
 &= \frac{1}{\hbar} \sqrt{\frac{a}{2\pi}} \left[\int_{-\infty}^0 e^{ap/\hbar} e^{ipx/\hbar} dp + \int_0^{\infty} e^{-ap/\hbar} e^{ipx/\hbar} dp \right]
 \end{aligned}$$

then, we apply a change of variable, $p \rightarrow -p$ for the first integral, and swap the limits:

$$\begin{aligned}
 \psi(x) &= \frac{1}{\hbar} \sqrt{\frac{a}{2\pi}} \left[\int_0^{\infty} e^{-ap/\hbar} e^{-ipx/\hbar} dp + \int_0^{\infty} e^{-ap/\hbar} e^{ipx/\hbar} dp \right] \\
 &= \frac{1}{\hbar} \sqrt{\frac{a}{2\pi}} \int_0^{\infty} e^{-ap/\hbar} \left(e^{-ipx/\hbar} + e^{ipx/\hbar} \right) dp \\
 &= \frac{1}{\hbar} \sqrt{\frac{a}{2\pi}} \int_0^{\infty} e^{-ap/\hbar} 2 \cos(px/\hbar) dp \\
 &= \frac{1}{\hbar} \sqrt{\frac{2a}{\pi}} \int_0^{\infty} \cos(px/\hbar) e^{-ap/\hbar} dp \\
 &= \frac{1}{\hbar} \sqrt{\frac{2a}{\pi}} \left(\frac{x \sin\left(\frac{px}{\hbar}\right) - a \cos\left(\frac{px}{\hbar}\right)}{a^2 + x^2} \hbar e^{-ap/\hbar} \right) \Big|_0^{\infty} \\
 \psi(x) &= \sqrt{\frac{2}{\pi}} \frac{a^{3/2}}{a^2 + x^2}
 \end{aligned}$$

Normalization check:

$$\begin{aligned}
 \int_{-\infty}^{\infty} |\psi(x)|^2 dx &= \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{a^3}{(a^2 + x^2)^2} dx \\
 &= \frac{2}{\pi} \frac{1}{2} \left(\frac{ax}{a^2 + x^2} + \tan^{-1}(x/a) \right) \Big|_{-\infty}^{\infty} \\
 &= \frac{1}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) \\
 &= 1
 \end{aligned}$$

Answer 3:

Both $\psi_p(p)$ and $\psi(x)$ are symmetric functions thus $\langle x \rangle = 0$, $\langle p \rangle = 0$.

$$\begin{aligned}
 \langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 |\psi(x)|^2 dx \\
 &= \frac{2}{\pi} a^3 \int_{-\infty}^{\infty} \frac{x^2}{(a^2 + x^2)^2} dx \\
 &= \frac{2a^3}{\pi} \frac{1}{2} \left(\frac{\tan^{-1}(x/a)}{a} - \frac{x}{a^2 + x^2} \right) \Big|_{-\infty}^{\infty} \\
 &= \frac{a^3}{\pi} \left(\frac{\pi}{2a} - \left(-\frac{\pi}{2a} \right) \right) \\
 &= a^2
 \end{aligned}$$

Thus, the uncertainty in position is:

$$\begin{aligned}
 \Delta x &= \sqrt{\langle x^2 \rangle + \langle x \rangle^2} \\
 &= \sqrt{a^2 - 0^2} \\
 &= a
 \end{aligned}$$

And,

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} p^2 |\psi_p(p)|^2 dp$$

$$\begin{aligned}
&= \frac{a}{\hbar} \int_{-\infty}^{\infty} p^2 e^{-2a|p|/\hbar} dp \\
&= \frac{2a}{\hbar} \int_0^{\infty} p^2 e^{-2ap/\hbar} dp \\
&= \frac{2a}{\hbar} \frac{1}{4} \left[-e^{-2ap/\hbar} \left(\frac{2p^2 a^2}{\hbar^2} + \frac{2ap}{\hbar} + 1 \right) \frac{\hbar^3}{a^3} \right] \Big|_0^{\infty} \\
&= \frac{\hbar^2}{2a^2}
\end{aligned}$$

Thus, the uncertainty in momentum is:

$$\begin{aligned}
\Delta p &= \sqrt{\langle p^2 \rangle + \langle p \rangle^2} \\
&= \sqrt{\frac{\hbar^2}{2a^2} - 0^2} \\
&= \frac{\hbar}{\sqrt{2}a}
\end{aligned}$$

So, we get:

$$\Delta x \cdot \Delta p = a \cdot \frac{\hbar}{\sqrt{2}a} = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}$$

which is in agreement with the uncertainty principle.

Answer 4:

This is quite similar to the infinite square well. The wave function inside the well is given by

$$\psi(x) = C \cos(kx) \quad \text{or} \quad \psi(x) = C \sin(kx)$$

where

$$k = \sqrt{\frac{2m(E - V(x))}{\hbar^2}} = \sqrt{\frac{2m(E + V_0)}{\hbar^2}}$$

The only difference between this wave number and the one for the infinite square well is: $E \rightarrow E + V_0$. Thus, we simply write down the energy as follows:

$$\underbrace{E = \frac{\hbar^2 n^2 \pi^2}{2mL^2}}_{\text{infinite square well}} \longrightarrow \underbrace{E + V_0 = \frac{\hbar^2 n^2 \pi^2}{2mL^2}}_{\text{this problem}}$$

Thus,

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2} - V_0$$

For $E_n < 0$:

$$\begin{aligned}
E_n &< 0 \\
\frac{\hbar^2 n^2 \pi^2}{2mL^2} - V_0 &< 0 \\
n^2 &< \frac{2mL^2 V_0}{\hbar^2 \pi^2} \\
n &< \frac{L}{\hbar c \pi} \sqrt{2mc^2 V_0} \\
n &< \frac{0.2 \text{ nm}}{197 \text{ eV} \cdot \text{nm}} \pi \sqrt{2 \times (0.511 \times 10^6 \text{ eV}) \times (20 \text{ eV})} \\
n &< 1.5
\end{aligned}$$

Thus, n can only be 1, the ground state.

Answer 5:

The wave function is given:

$$\begin{aligned}
\psi(x) &= C \left(\alpha^{3/2} x^3 - \frac{3}{4} \sqrt{\alpha} x \right) e^{-\alpha x^2} \\
&= \frac{C \sqrt{\alpha}}{4} [4\alpha x^3 - 3x] e^{-\alpha x^2}
\end{aligned}$$

The first partial derivative of the wave function:

$$\begin{aligned}\frac{\partial}{\partial x}\psi(x) &= \frac{C\sqrt{\alpha}}{4} \cdot e^{-\alpha x^2} \cdot [12\alpha x^2 - 3 - 2\alpha x(4\alpha x^3 - 3x)] \\ &= -\frac{C\sqrt{\alpha}}{4} \cdot e^{-\alpha x^2} \cdot (8\alpha^2 x^4 - 18\alpha x^2 + 3)\end{aligned}$$

The second partial derivative of the wave function:

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\psi(x) &= -\frac{C\sqrt{\alpha}}{4} \cdot e^{-\alpha x^2} \cdot [32\alpha^2 x^3 - 36\alpha x - 2\alpha x(8\alpha^2 x^4 - 18\alpha x^2 + 3)] \\ &= \frac{C\alpha^{3/2}}{2} [8\alpha^2 x^4 - 34\alpha x^2 + 21] \cdot x \cdot e^{-\alpha x^2} \\ &= \frac{C\alpha^{3/2}}{2} (4\alpha x^2 - 3)(2\alpha x^2 - 7) \cdot x \cdot e^{-\alpha x^2} \\ &= \underbrace{\frac{C\sqrt{\alpha}}{4} [4\alpha x^3 - 3x] e^{-\alpha x^2}}_{\psi(x)} \cdot 2\alpha(2\alpha x^2 - 7) \\ \frac{\partial^2}{\partial x^2}\psi(x) &= [2\alpha(2\alpha x^2 - 7)] \psi(x)\end{aligned}$$

Before writing the Schrödinger equation, let us write the potential function in terms of α :

$$V(x) = \frac{1}{2}m\omega^2 x^2 \quad \alpha \equiv \frac{m\omega}{2\hbar} \quad V(x) = \frac{\hbar^2}{m} 2\alpha^2 x^2$$

Then, the Schrödinger equation becomes:

$$\begin{aligned}E\psi(x) &= \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)\right) \psi(x) \\ E\psi(x) &= \left(-\frac{\hbar^2}{2m} [2\alpha(2\alpha x^2 - 7)] + V(x)\right) \psi(x) \\ E\psi(x) &= \left(-\frac{\hbar^2}{2m} [2\alpha(2\alpha x^2 - 7)] + \frac{\hbar^2}{m} 2\alpha^2 x^2\right) \psi(x) \\ E &= \frac{\hbar^2}{m} (2\alpha^2 x^2 - 2\alpha^2 x^2) + \frac{7\hbar^2 \alpha}{m} \\ E &= \frac{7\hbar^2 \alpha}{m}\end{aligned}$$

Both sides of this equation are constants, thus this wave function satisfies the Schrödinger equation for the simple harmonic oscillator potential.

Now, we can substitute $\alpha \equiv \frac{m\omega}{2\hbar}$:

$$E = \frac{7\hbar^2}{m} \frac{m\omega}{2\hbar} = \frac{7}{2}\hbar\omega$$

which gives the energy for the third excited state ($n = 4$): $E_n = (n - 1/2)\hbar\omega$