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Problem Set #6 Solutions

Answer 1:

The force due to the given potential is:

$$\vec{F}(\vec{r}) = -\vec{\nabla}V = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{r} = -\frac{3}{2}\frac{a}{r^{5/2}}\hat{r}$$

For a circular orbit, the centripetal force is equal to this:

$$\begin{aligned} -\frac{3}{2}\frac{a}{r^{5/2}}\hat{r} &= -\frac{mv^2}{r}\hat{r} \\ \frac{1}{2}mv^2 &= \frac{3}{4}\frac{a}{r^{3/2}} \\ E_k &= -\frac{3}{4}V \qquad (1) \end{aligned}$$

Then, the total energy is

$$E = V + E_k = V - \frac{3}{4}V = \frac{V}{4} \qquad (2)$$

The angular momentum is quantized as follows:

$$L = mvr = n\hbar \qquad \to \qquad v = \frac{n\hbar}{mr}$$

Then, the kinetic energy can be written using the speed found above:

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{n\hbar}{mr}\right)^2 = \frac{n^2\hbar^2}{2mr^2}$$

We combine this with the kinetic energy found in equation (1):

$$\begin{aligned} -\frac{3}{4}V &= \frac{n^2\hbar^2}{mr^2} \\ \frac{3}{4}\frac{a}{r^{3/2}} &= \frac{n^2\hbar^2}{2mr^2} \\ r &= \frac{4n^2\hbar^4}{9m^2a^2} \end{aligned}$$

Then, we use the expression for r to find the total energy shown in equation (2):

$$E_n = \frac{V}{4} = -\frac{1}{4}\frac{a}{r^{3/2}} = -\frac{27}{32}\frac{a^4m^3}{\hbar^6}\frac{1}{n^6}$$

Answer 2:

The potential energy for such a system would be:

$$V(\vec{r}) = \frac{kq_1q_2}{r} = \frac{k(Ze)(-e)}{r} = -\frac{Zke^2}{r}$$

At this point, this system is identical to the hydrogen atom for which the potential is $V(\vec{r}) = -\frac{ke^2}{r}$, with only one exception; the constants ke^2 are changed to Zke^2. Thus, we find the modified equations for this system by changing $\alpha^2 \to Z^2\alpha^2$ ($\alpha \equiv \frac{ke^2}{\hbar c}$) without further work:

$$E_n = -\frac{Z^2\alpha^2mc^2}{2}\frac{1}{n^2}$$

Transition from $n_i = 3$ to $n_f = 1$:

$$E_\gamma = E_3 - E_1 = -\frac{Z^2\alpha^2 mc^2}{2}\left(\frac{1}{3^2} - \frac{1}{1^2}\right) = \frac{4}{9}(Z^2\alpha^2 mc^2)$$

The photon energy: $E_\gamma = hc/\lambda$, thus the wavelength is expressed by:

$$\lambda = \frac{9}{4}\frac{hc}{Z^2\alpha^2 mc^2}$$

where $Z = 8$ for stripped Oxygen, and m is the mass of the electron. If we put all the numbers in this expression, we find the wavelength:

$$\lambda = 1.6 \text{ nm}$$

$Z = 1$ for a hydrogen atom. Thus this wavelength is $1/Z^2 = 1/64^{th}$ of the corresponding wavelength for an ordinary hydrogen atom.

Answer 3:

The deBroglie wavelength of a particle with speed v is

$$\lambda = \frac{h}{mv}$$

Then, the average deBroglie wavelength is

$$\langle\lambda\rangle = \left\langle\frac{h}{mv}\right\rangle = \frac{h}{m}\left\langle\frac{1}{v}\right\rangle$$

Note that:

$$\left\langle\frac{1}{v}\right\rangle \neq \frac{1}{\langle v\rangle} \neq \frac{1}{\sqrt{\langle v^2\rangle}}$$

(Try taking the above averages for $v_1 = 1$ and $v_2 = 3$ to confirm this.) Thus, you cannot use $\langle E\rangle = \frac{3}{2}kT = \frac{1}{2}m\langle v^2\rangle$ to extract $\langle 1/v\rangle$. We need the probability distribution as a function of v for an ideal monatomic gas in order to find the average $1/v$; it is given by the Maxwell distribution (Phys 102, Phys 221):

$$F(v)\,\mathrm{d}v = C\,v^2 \exp\left(\frac{\frac{1}{2}mv^2}{kT}\right)\mathrm{d}v$$

where C is the normalization constant. Then, the average could be find as:

$$\begin{aligned}
\langle 1/v\rangle &= \frac{\int_0^\infty (1/v)P(v)\,\mathrm{d}v}{\int_0^\infty P(v)\,\mathrm{d}v} \\
&= \frac{C\int_0^\infty \frac{1}{v}v^2\, e^{-\frac{m}{2kT}v^2}\mathrm{d}v}{C\int_0^\infty v^2\, e^{-\frac{m}{2kT}v^2}\mathrm{d}v} \\
&= \frac{\int_0^\infty v\, e^{-yv^2}\mathrm{d}v}{\int_0^\infty v^2\, e^{-yv^2}\mathrm{d}v} \qquad \text{where } y \equiv m/2kT \\
&= \left(\frac{1}{2y}\right)\Big/\left(\frac{\sqrt{\pi}}{4y^{3/2}}\right) = \sqrt{\frac{4y}{\pi}} \\
&= \sqrt{\frac{2m}{\pi kT}}
\end{aligned}$$

Then, the average wavelength is

$$\langle\lambda\rangle = \frac{h}{m}\left\langle\frac{1}{v}\right\rangle = \sqrt{\frac{2h^2}{\pi kTm}}$$

At room temperature, that becomes about 2 Å, about the size of a Hydrogen atom (0.5 Å).

Answer 4:

The deBroglie wavelength of the electron is

$$\lambda = \frac{h}{p}$$

We are asked to find the speed for two different wavelengths, one of which is many orders of magnitude smaller than the other, thus at least one of the calculation should require relativistic momentum equation. So, let us start with the relativistic equation instead of $p = mv$.

$$
\begin{aligned}
\lambda &= \frac{hc}{pc} \\
\frac{hc}{\lambda} &= pc \\
(hc/\lambda)^2 &= (pc)^2 \\
&= E^2 - (mc^2)^2 \\
&= (mc^2)^2\,\gamma^2 - (mc^2)^2 \\
\frac{(hc/\lambda)^2}{(mc^2)^2} &= \gamma^2 - 1 \\
(h/\lambda mc)^2 &= \frac{1}{1-\beta^2} - 1 \\
&= \frac{\beta^2}{1-\beta^2} \\
(\lambda mc/h)^2 &= 1/\beta^2 - 1 \\
\beta^2 &= \frac{1}{(\lambda mc/h)^2 + 1} \\
v &= \frac{c}{\sqrt{(\lambda mc/h)^2 + 1}}
\end{aligned}
$$

Using this formula, we find

a) For $\lambda = 30$ cm $= 0.3$ m:

$$v = 2.4 \times 10^{-3} \text{ m/s}$$

b) For $\lambda = 30$ fm $= 30 \times 10^{-15}$ m:

$$v \approx 2.99 \times 10^{8} \text{ m/s} \approx c$$

Thus, it is more appropiate to give $\gamma = \left(\frac{(hc/\lambda)^2}{(mc^2)^2} + 1\right)^{1/2} \approx \frac{hc/\lambda}{mc^2} \approx 80$

Answer 5:

The deBroglie wavelength is $\lambda = h/p$. If the wavelengths for the γ and μ are the same, then the momentums of these two particles must be the same: $p_\gamma = p_\mu$.

$$\begin{aligned}
p_\gamma &= p_\mu \\
(p_\gamma c)^2 &= (p_\mu c)^2 \\
(p_\gamma c)^2 &= E_\mu^2 - (m_\mu c^2)^2 \\
E_\gamma^2 &= E_\mu^2 - (m_\mu c^2)^2 \\
&= 200^2 - 106^2 \text{ MeV}^2 \\
E_\gamma &= 170 \text{ MeV}
\end{aligned}$$