Boğaziçi University
Department of Physics

Phys 311/407 Summer 2015

Problem Set #7 Solutions

Answer 1:

a) We normalize the probability distribution:

$$\begin{aligned} 1 &= \int \mathrm{d}P \\ &= \int_{-\infty}^{\infty} |\psi(x)|^2 \mathrm{d}x \\ &= \int_{-\infty}^{\infty} |C|^2 \, e^{-(x/\sigma)^2} \mathrm{d}x \\ &= |C|^2 \sqrt{\sigma^2 \pi} \\ |C| &= (\sigma^2 \, \pi)^{-1/4} \end{aligned}$$

Take the phase of C to be zero ($C = |C| \cdot e^{i\phi=0} = |C|$). Thus,

$$\psi(x) = (\sigma^2 \, \pi)^{-1/4} e^{\frac{1}{2}(x/\sigma)^2}$$

b) The wave number distribution function, $g(k)$, is the Fourier transformation of the position wave function:

$$\begin{aligned} g(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x) e^{ikx} \, \mathrm{d}x \\ &= (4\sigma^2 \pi^3)^{-1/4} \int_{-\infty}^{\infty} e^{-[(\frac{1}{2\sigma^2})x^2 + (-ik)x]} \, \mathrm{d}x \end{aligned}$$

Then, we define $a \equiv \frac{1}{2\sigma^2}$, $b \equiv -ik$ in order to use the integral provided by the problem set:

$$g(k) = (4\sigma^2 \pi^3)^{-1/4} \sqrt{\frac{\pi}{a}} e^{b^2/4a}$$

Then, we replaced a and b by their definitions given above:

$$g(k) = (\sigma^2/\pi)^{1/4} \, e^{-\frac{1}{2}\sigma^2 k^2}$$

c)

$$\langle x \rangle = \int \mathrm{d}P \cdot x = \int_{-\infty}^{\infty} x \cdot |\psi(x)|^2 \mathrm{d}x = C^2 \int_{-\infty}^{\infty} x \cdot e^{-(x/\sigma)^2} \mathrm{d}x$$

The integrant is an odd function, and the integral limits are symmetric, thus the integral vanishes:

$$\langle x \rangle = 0$$

d)

$$\left\langle x^2 \right\rangle = \int \mathrm{d}P \cdot x^2 = \int_{-\infty}^{\infty} x^2 \cdot |\psi(x)|^2 \mathrm{d}x = C^2 \int_{-\infty}^{\infty} x^2 \cdot e^{-(x/\sigma)^2} \mathrm{d}x = C^2 \left(\frac{\sigma^3}{2} \sqrt{\pi} \right)$$

We insert the normalization constant C found in part (a):

$$\left\langle x^2 \right\rangle = \frac{1}{\sqrt{\sigma^2 \pi}} \frac{\sigma^3 \sqrt{\pi}}{2} = \frac{\sigma^2}{2}$$

e)

$$(\Delta x)^2 = \int P \cdot (x - \langle x \rangle)^2 = \int P \cdot (x - 0)^2 = \int P \cdot x^2 = \left\langle x^2 \right\rangle$$

Thus

$$\Delta x = \frac{\sigma}{\sqrt{2}}$$

f) We compare the forms of $\psi(x)$ and $g(k)$. They are both Gaussian functions with different width and amplitude. The amplitude (normalization) should not effect the uncertainty, thus all we care is the width of the Gaussian, so we compare $e^{-\frac{1}{2}(\frac{x}{\sigma})^2} \leftrightarrow e^{-\frac{1}{2}(\frac{k}{1/\sigma})^2}$. Then, replacing σ in the Δx expression with $1/\sigma$ should give us Δk:

$$\Delta k = \frac{1}{\sigma \sqrt{2}}$$

The relation between k and p is $p = \hbar\, k$:

$$\Delta p_x = \hbar \Delta k = \frac{\hbar}{\sigma \sqrt{2}}$$

g)

$$\Delta x \cdot \Delta p_x = \frac{\sigma}{\sqrt{2}} \frac{\hbar}{\sigma \sqrt{2}} = \hbar/2$$

Which is the lower limit for the uncertainty principle. For any other wave function this quantity is greater than $\hbar/2$.

Answer 2:

The musicians with "perfect pitch" are those who can identify the notes without any reference notes. However, distinguishing a note (call it "frequency" if you wish) from another is always done with comparison. This comparison is done w.r.t. another source or their internal senses (humming is one way to do this). Comparison of two notes, on the other hand depends on the difference between the frequencies. Remember that the interference between f_1 and f_2 creates beats with a frequency of $|f_1 - f_2|$, so one should be able to measure a frequency of at the order of $|f_1 - f_2|$ (the envelope frequency is half of that). If the duration of the measurement/listening of the sound is too short compared to the period of $f = |f_1 - f_2|$, then the measurement can not be accurate. In fact, one needs more than one period to determine the beat period. And, since we are talking about small frequency differences, the period is long, sometimes seconds. Thus she may not determine the note precisely if the duration of the observation, $\Delta t < \frac{1}{|f_1 - f_2|}$. Another way of saying this is:

$$\Delta t \cdot \Delta f > 1$$

for a successful measurement. You may call this the *uncertainty principle* for the frequency measurement.

Answer 3:

This is a discrete system, the averaging operation $\langle M \rangle = \int M \cdot \mathrm{d}P$ becomes $\langle M \rangle = \sum_n M_n P_n$. Where, $x_1 = -L$ with $P_1 = 0.5$ and $x_2 = L$ with $P_2 = 0.5$. Then, the average x:

$$\langle x \rangle = \sum_{n=1}^{2} x_n P_n = \frac{1}{2}(-L) + \frac{1}{2}(L) = 0$$

The average x^2:

$$\left\langle x^2 \right\rangle = \sum_{n=1}^{2} x_n^2 P_n = \frac{1}{2}(-L)^2 + \frac{1}{2}(L)^2 = L^2$$

The position uncertainty:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{L^2 - 0^2} = L$$

Answer 4:

We are asked to give a qualitative answer. Let us assume that the box is cubical with a side dimension of $L = 10^{-10}$ m. In the ground state, the deBroglie wavelength should be about the size of the box; L. Then, the momentum, $p_{\min} c = hc/\lambda \sim h/L \sim 10$ keV for both particles – note also that this is non-relativistic, and that is the minimum momentum they can have; shorter wavelengths with $\frac{\lambda}{2}n = L$ will lead larger momentum according to the deBroglie wavelength formula. The kinetic energy for a non-relativistic particle is $E_{k,\min} = p_{\min}^2/2m$, thus the lighter particle will have a larger minimum kinetic energy: The answer is electron.