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Problem Set #8 Solutions

Answer 1:

We write the normalization equation:

$$1 = \int_{-\infty}^{\infty} |\psi(p)|^2 \,\mathrm{d}p$$

Since, the wave function includes an absolute value function, we have to divide the integrant into two parts:

$$1 = \int_{-\infty}^{0} \left[Ce^{ap/\hbar}\right]^2 \mathrm{d}p + \int_{0}^{\infty} \left[Ce^{-ap/\hbar}\right]^2 \mathrm{d}p$$

By symmetry, both integrals are equal to each other, thus we evaluate one of them and multiply it by two:

$$\begin{aligned}
1 &= 2 \cdot C^2 \int_0^{\infty} e^{-2ap/\hbar}\,\mathrm{d}p \\
&= 2C^2 \left(-\frac{\hbar}{2a}\right)\left(e^{-2ap/\hbar}\right)\Big|_0^{\infty} \\
&= -\frac{C^2\hbar}{a}(0-1) \\
C &= \sqrt{\frac{a}{\hbar}}
\end{aligned}$$

Answer 2:

Note that, we derived the fourier transformation equations in class between $\psi(x) \leftrightarrow g(k)$. Using $p = \hbar k$, $\mathrm{d}p = \hbar\,\mathrm{d}k$, we can:

$$\begin{aligned}
\mathrm{d}P &= |g(k)|^2\,\mathrm{d}k \\
&= |g(p/\hbar)|^2 \frac{\mathrm{d}p}{\hbar} \\
|\psi_p(p)|^2\,\mathrm{d}p &= |g(p/\hbar)|^2 \frac{\mathrm{d}p}{\hbar}
\end{aligned}$$

which gives us:

$$\psi_p(p) = \frac{1}{\sqrt{\hbar}}\, g(p/\hbar)$$

or, equivalently:

$$g(k) = \sqrt{\hbar}\,\psi_p(k\hbar)$$

[Note that, we ignored the phase, $e^{i\phi}$, while dealing with the magnitude function above. This is no concern for us, as it will satisfy the Scrödinger equation with or without a phase, which is a constant .]

Now, we can rewrite our Fourier transformation using momentum wave function:

$$\begin{aligned}
\psi(x) &= \frac{1}{\sqrt{2\pi}} \int g(k) e^{ikx}\,\mathrm{d}k \\
\psi(x) &= \frac{1}{\sqrt{2\pi}} \int \sqrt{\hbar}\,\psi_p(k\hbar) e^{ipx/\hbar}\,\frac{\mathrm{d}p}{\hbar}
\end{aligned}$$

Thus, we get:

$$\boxed{\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi_p(p) e^{ipx/\hbar}\,\mathrm{d}p} \quad \text{similarly} \rightarrow \quad \boxed{\psi_p(p) = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(x) e^{-ipx/\hbar}\,\mathrm{d}x}$$

Now, we can perform the transformation:

$$\begin{aligned}
\psi(x) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\psi_p(p)e^{ipx/\hbar}dp \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{a}{\hbar}}\int_{-\infty}^{\infty}e^{-a|p|/\hbar}e^{ipx/\hbar}dp \\
&= \frac{1}{\hbar}\sqrt{\frac{a}{2\pi}}\left[\int_{-\infty}^{0}e^{ap/\hbar}e^{ipx/\hbar}\,\mathrm{d}p+\int_{0}^{\infty}e^{-ap/\hbar}e^{ipx/\hbar}\,\mathrm{d}p\right]
\end{aligned}$$

then, we apply a change of variable, $p \to -p$ for the first integral, and swap the limits:

$$\begin{aligned}
\psi(x) &= \frac{1}{\hbar}\sqrt{\frac{a}{2\pi}}\left[\int_{0}^{\infty}e^{-ap/\hbar}e^{-ipx/\hbar}\,\mathrm{d}p+\int_{0}^{\infty}e^{-ap/\hbar}e^{ipx/\hbar}\,\mathrm{d}p\right] \\
&= \frac{1}{\hbar}\sqrt{\frac{a}{2\pi}}\int_{0}^{\infty}e^{-ap/\hbar}\left(e^{-ipx/\hbar}+e^{ipx/\hbar}\right)\mathrm{d}p \\
&= \frac{1}{\hbar}\sqrt{\frac{a}{2\pi}}\int_{0}^{\infty}e^{-ap/\hbar}\,2\cos(px/\hbar)\,\mathrm{d}p \\
&= \frac{1}{\hbar}\sqrt{\frac{2a}{\pi}}\int_{0}^{\infty}\cos(px/\hbar)\,e^{-ap/\hbar}\,\mathrm{d}p \\
&= \frac{1}{\hbar}\sqrt{\frac{2a}{\pi}}\left(\frac{x\sin\left(\frac{px}{\hbar}\right)-a\cos\left(\frac{px}{\hbar}\right)}{a^2+x^2}\hbar\,e^{-ap/\hbar}\right)\Bigg|_{0}^{\infty} \\
\psi(x) &= \sqrt{\frac{2}{\pi}}\frac{a^{3/2}}{a^2+x^2}
\end{aligned}$$

Normalization check:

$$\begin{aligned}
\int_{-\infty}^{\infty}|\psi(x)|^2\,\mathrm{d}x &= \frac{2}{\pi}\int_{-\infty}^{\infty}\frac{a^3}{(a^2+x^2)^2}\,\mathrm{d}x \\
&= \frac{2}{\pi}\frac{1}{2}\left(\frac{ax}{a^2+x^2}+\tan^{-1}(x/a)\right)\Bigg|_{-\infty}^{\infty} \\
&= \frac{1}{\pi}\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) \\
&= 1
\end{aligned}$$

Answer 3:

Both $\psi_p(p)$ and $\psi(x)$ are symmetric functions thus $\langle x\rangle = 0$, $\langle p\rangle = 0$.

$$\begin{aligned}
\left\langle x^2\right\rangle &= \int_{-\infty}^{\infty}x^2\,|\psi(x)|^2\,\mathrm{d}x \\
&= \frac{2}{\pi}a^3\int_{-\infty}^{\infty}\frac{x^2}{(a^2+x^2)^2}\,\mathrm{d}x \\
&= \frac{2a^3}{\pi}\frac{1}{2}\left(\frac{\tan^{-1}(x/a)}{a}-\frac{x}{a^2+x^2}\right)\Bigg|_{-\infty}^{\infty} \\
&= \frac{a^3}{\pi}\left(\frac{\pi}{2a}-\left(-\frac{\pi}{2a}\right)\right) \\
&= a^2
\end{aligned}$$

Thus, the uncertainty in position is:

$$\begin{aligned}
\Delta x &= \sqrt{\langle x^2\rangle+\langle x\rangle^2} \\
&= \sqrt{a^2-0^2} \\
&= a
\end{aligned}$$

And,

$$\left\langle p^2\right\rangle = \int_{-\infty}^{\infty}p^2\,|\psi_p(p)|^2\,\mathrm{d}p$$

$$
\begin{aligned}
&= \frac{a}{\hbar}\int_{-\infty}^{\infty} p^2\, e^{-2a|p|/\hbar}\,\mathrm{d}p \\
&= \frac{2a}{\hbar}\int_{0}^{\infty} p^2\, e^{-2ap/\hbar}\,\mathrm{d}p \\
&= \frac{2a}{\hbar}\frac{1}{4}\left[-e^{-2ap/\hbar}\left(\frac{2p^2a^2}{\hbar^2}+\frac{2ap}{\hbar}+1\right)\frac{\hbar^3}{a^3}\right]\Bigg|_0^{\infty} \\
&= \frac{\hbar^2}{2a^2}
\end{aligned}
$$

Thus, the uncertainty in momentum is:

$$
\begin{aligned}
\Delta p &= \sqrt{\langle p^2\rangle + \langle p\rangle^2} \\
&= \sqrt{\frac{\hbar^2}{2a^2} - 0^2} \\
&= \frac{\hbar}{\sqrt{2}a}
\end{aligned}
$$

So, we get:

$$\Delta x \cdot \Delta p = a \cdot \frac{\hbar}{\sqrt{2}a} = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}$$

which is in agreement with the uncertainty principle.

Answer 4:

This is quite similar to the infinite square well. The wave function inside the well is given by

$$\psi(x) = C\cos(kx) \qquad \text{or} \qquad \psi(x) = C\sin(kx)$$

where

$$k = \sqrt{\frac{2m(E - V(x))}{\hbar^2}} = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}$$

The only difference between this wave number and the one for the infinite square well is: $E \rightarrow E + V_0$. Thus, we simply write down the energy as follows:

$$\underbrace{E = \frac{\hbar^2 n^2 \pi^2}{2mL^2}}_{\text{infinite square well}} \qquad \longrightarrow \qquad \underbrace{E + V_0 = \frac{\hbar^2 n^2 \pi^2}{2mL^2}}_{\text{this problem}}$$

Thus,

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2} - V_0$$

For $E_n < 0$:

$$
\begin{aligned}
E_n &< 0 \\
\frac{\hbar^2 n^2 \pi^2}{2mL^2} - V_0 &< 0 \\
n^2 &< \frac{2mL^2 V_0}{\hbar^2 \pi^2} \\
n &< \frac{L}{\hbar c \pi}\sqrt{2mc^2 V_0} \\
n &< \frac{0.2\ \text{nm}}{197\ \text{eV}\cdot\text{nm}\,\pi}\sqrt{2 \times (0.511 \times 10^6\ \text{eV}) \times (20\ \text{eV})} \\
n &< 1.5
\end{aligned}
$$

Thus, n can only be 1, the ground state.

Answer 5:

The wave function is given:

$$
\begin{aligned}
\psi(x) &= C\left(\alpha^{3/2}x^3 - \frac{3}{4}\sqrt{\alpha}x\right)e^{-\alpha x^2} \\
&= \frac{C\sqrt{\alpha}}{4}\left[4\alpha x^3 - 3x\right]e^{-\alpha x^2}
\end{aligned}
$$

The first partial derivative of the wave function:

$$\begin{aligned}\frac{\partial}{\partial x}\psi(x) &= \frac{C\sqrt{\alpha}}{4}\cdot e^{-\alpha x^2}\cdot\left[12\alpha x^2-3-2\alpha x(4\alpha x^3-3x)\right]\\ &= -\frac{C\sqrt{\alpha}}{4}\cdot e^{-\alpha x^2}\cdot\left(8\alpha^2x^4-18\alpha x^2+3\right)\end{aligned}$$

The second partial derivative of the wave function:

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\psi(x) &= -\frac{C\sqrt{\alpha}}{4}\cdot e^{-\alpha x^2}\cdot\left[32\alpha^2x^3-36\alpha x-2\alpha x\left(8\alpha^2x^4-18\alpha x^2+3\right)\right]\\ &= \frac{C\alpha^{3/2}}{2}\left[8\alpha^2x^4-34\alpha x^2+21\right]\cdot x\cdot e^{-\alpha x^2}\\ &= \frac{C\alpha^{3/2}}{2}(4\alpha x^2-3)(2\alpha x^2-7)\cdot x\cdot e^{-\alpha x^2}\\ &= \underbrace{\frac{C\sqrt{\alpha}}{4}\left[4\alpha x^3-3x\right]e^{-\alpha x^2}}_{\psi(x)}\cdot 2\alpha(2\alpha x^2-7)\\ \frac{\partial^2}{\partial x^2}\psi(x) &= \left[2\alpha(2\alpha x^2-7)\right]\psi(x)\end{aligned}$$

Before writing the Schrödinger equation, let us write the potential function in terms of α:

$$V(x)=\frac{1}{2}m\omega^2x^2 \quad \xrightarrow{\alpha\equiv\frac{m\omega}{2\hbar}} \quad V(x)=\frac{\hbar^2}{m}2\alpha^2x^2$$

Then, the Schrödinger equation becomes:

$$\begin{aligned}E\,\psi(x) &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V(x)\right)\psi(x)\\ E\,\psi(x) &= \left(-\frac{\hbar^2}{2m}\left[2\alpha(2\alpha x^2-7)\right]+V(x)\right)\psi(x)\\ E\,\psi(x) &= \left(-\frac{\hbar^2}{2m}\left[2\alpha(2\alpha x^2-7)\right]+\frac{\hbar^2}{m}2\alpha^2x^2\right)\psi(x)\\ E &= \frac{\hbar^2}{m}(2\alpha^2x^2-2\alpha^2x^2)+\frac{7\hbar^2\alpha}{m}\\ E &= \frac{7\hbar^2\alpha}{m}\end{aligned}$$

Both sides of this equation are constants, thus this wave function satisfies the Schrödinger equation for the simple harmonic oscillator potential.

Now, we can substitute $\alpha\equiv\frac{m\omega}{2\hbar}$:

$$E=\frac{7\hbar^2}{m}\frac{m\omega}{2\hbar}=\frac{7}{2}\hbar\omega$$

which gives the energy for the third exited state ($n=4$): $E_n=(n-1/2)\hbar\omega$